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"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

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On the Arithmometer of M. Thomas (de Colmar), and its application to the Construction of Life Contingency Tables. By PETER GRAY, F.R.A.S., F.R.M.S., *Honorary Member of the Institute of Actuaries.*

THE Arithmometer of M. Thomas (de Colmar) has been already brought under the notice of the readers of this *Journal* by General Hannyngton, in a remarkably lucid and suggestive paper, which will be found at p. 244, vol. xvi. General Hannyngton, in his paper, explains the manner of working the machine, and gives examples of some of its applications to the construction of actuarial tables, with hints as to others. These afford an idea of the very striking adaptation of the machine to the formation of such tables; and they cannot fail to have excited the interest of many of the readers.

The present paper is intended to be supplemental to that of General Hannyngton, and in it the adaptation in question will be further shown. An attempt will also be made to systematize the manner of its application, and detailed examples in illustration will be given. There will be no need to say anything here as to the actual working of the machine, since this has been so well explained by General Hannyngton. There are, nevertheless, certain points in the manipulation to which it may be well to advert in the outset.

It is usual to describe the Arithmometer as a machine which enables a person, however unskilled himself, to perform the operations of multiplication and division with facility, rapidity, and unfailing accuracy. This, as a description, is correct as far as it goes; but as an enumeration of the properties of the machine, it is inadequate and defective. It entirely omits that property which forms its special adaptation to our purpose, and in default of which its utility would be comparatively limited. Besides the facilitation of the operations named, the machine will also, in forming the product of two given numbers, either add that product to, or subtract it from, another given number, according to the pleasure of the operator. Abundant illustration of the application of this property* will be found in the present paper.

There are three forms, then, to the numerical evaluation of which the Arithmometer is directly applicable. These may be symbolized as follows:—

$$QR, \quad \frac{Q}{R}, \quad \text{and } P \pm QR;$$

P, Q and R denoting any given numbers.

* An illustration of it is given by General Hannyngton, in the formation of $D_{x.y}$.

Familiar instances of these forms, in the present connexion, are:—

$$\text{QR} \quad . \quad . \quad . \quad . \quad . \quad . \quad D_x = l_x v^x;$$

$$\frac{\text{Q}}{\text{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad a_x = \frac{N_x}{D_x};$$

and

$$\begin{cases} \text{P}+\text{QR} \quad . \quad . \quad . \quad . \quad . \quad N_x = N_{x+1} + l_{x+1} v^{x+1}; \\ \text{P}-\text{QR} \quad . \quad . \quad . \quad . \quad . \quad A_x = 1-(1-v)(1+a_x). \end{cases}$$

The manner of applying the machine to the evaluation of expressions of the first and second forms; in other words, the manner of performing the operations of multiplication and division, has been sufficiently explained in General Hannyngton's paper. Of its application to the third form some further elucidation is necessary.

The special adaptation of this form to the construction of tables may be shown as follows:—

If u_x be any function of x, we always have,

$$u_{x+1} = u_x + \Delta u_x;$$

and this is of the form $\text{P} \pm \text{QR}$ if Δu_x, the difference of u_x, can be exhibited as the product of two given numbers. In such cases, then, (and it will soon appear that they are by no means rare), we have, in forming a series of successive values, the benefit of a continuous process, with its attendant advantages. The result of each operation becomes the P of the next, and we pass from a preceding to a succeeding value by adding to (or subtracting from) the former the product of two known numbers. And this, as we have seen, is a function to which the Arithmometer most readily lends itself.

In the applications of the machine there are thus, it appears, three numbers to be dealt with; and provision is accordingly made on it of three spaces for their separate exhibition. Two of these spaces are on the *slide*, and the third on the *face*. I propose to designate them by S_1, S_2, and F, respectively; and the numbers occupying the several spaces will be denoted by the same symbols, respectively distinguished, however, when so used, by being enclosed in parentheses, thus:—(S_1), (S_2), (F).

In employing the formula $\text{P} \pm \text{QR}$, the number P is placed on S_1, and Q (the *in* factor), upon F; multiplication being then made by R (the *out* factor), this number appears upon S_2, and the result, which is the required value of $\text{P} \pm \text{QR}$, replaces P upon S_1.

A point in regard to which difficulty is felt, in commencing the use of the Arithmometer, is the setting of the numbers P and Q upon the machine, so that the product QR shall, when formed, fall in its proper place with respect to P. The rule for this purpose (which has not hitherto been given) is very simple. It is:—

Draw out the slide as many holes as there are decimal places in R, the *out* factor; and place P and Q upon S_1 and F respectively, in such a manner as that like denominations shall stand under (and over) like. The slide is then to be pushed home; and, the multiplication being performed, the correct result will appear on S_1.

If $P=0$, that is, if it is only the product QR that is required, no preliminary drawing out of the slide is necessary.

So far as the results of the formulæ QR, $P\pm QR$ are concerned, it is obviously indifferent which of the two, Q, R, we employ as the *in* factor; in each case of practical application, however, there are usually circumstances sufficient to determine our choice in favour of one or the other. I shall in general, when necessary for distinction, let Q denote the *in* factor—the factor to be set upon F; and R, of course, to denote the *out* factor—the factor to be employed as a multiplier.

The points that have been now more specially adverted to, and others that may arise, will find ample illustration in the examples appended to the problems to which I now proceed.

The examples will be taken from *The Institute of Actuaries' Life Tables*, H^M mortality, and three per-cent interest.

Problem I.—Given a table of annuities; to form the corresponding table of assurances.

We have, $$A_x=1-(1-v)(1+a_x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad \Delta A_x=-(1-v)\Delta a_x;$$

whence, $$A_{x+1}=A_x+\Delta A_x=A_x-(1-v)\Delta a_x \quad . \quad . \quad . \quad . \quad (2)$$

By these formulæ the required table will be constructed, (1) coming into use for the formation of the initial term, and (2) for the continuous formation of the subsequent terms.

Example 1.—The given table of annuities is that on single lives, p. 14 (*Institute Tables*); and it is required to reproduce the table of assurances in the adjoining column.

Here we have, by (1),

$$P= \quad 1\cdot0000000,$$
$$Q=1-v = \cdot0291262,$$
$$R=1+a_{10}=25\cdot1484.$$

We here take $1-v$ for Q, because it remains constant during the process; and we use six significant figures because, there being also six in R, we desire to have our results true to the same extent.

P and Q are placed on the machine as follows:—

```
S₁  000010000000|0000
F       00291262|
```

There are *four* decimal places in R; *therefore*, by the rule, the denominations of the figures in P and Q are made to correspond when the slide is drawn out *four* holes.

One of the ivory pegs which accompany the machine is now inserted in the place of the decimal point, and another between the sixth and seventh decimal places; the last-named place also (the first beyond the peg) is increased by 5. The pegs serve to direct the eye to the part of the slide where the figures to be taken out for record are to be found; and the effect of the addition of 5 in the *next* place being to impart to the adjoining figure the usual correction, when necessary,* the eye, in taking out the result, has never to travel beyond the space embraced by the two pegs.

I repeat the representation of the setting of the numbers on the machine for the present example, adding indications of the positions of the pegs and of the increased figure:—

```
             *
00001,000000,5|0000
      00291262|
```

The same indications will be used in subsequent representations of the setting of the machine.

When the slide is pushed in as directed, the above setting will assume the following appearance:—

```
             *
00001,000000,50000|
          00291262|
```

The form here being P−QR, and both the factors Q, R, being positive, the regulator is set for subtraction; multiplication is then made by R=251484, and the result is,

```
               *
00000,267523,17192.
```

* The method here made use of for the correction of the last figure in the terms of series formed by addition (or subtraction) was, I believe, first suggested and exemplified by me, in a work published in 1849. Among the instances of its advantageous employment that have come to my knowledge, the most striking is one that occurred to Dr. Farr, in the construction of his tables of log v^x (*Tables of Life-Times*, pp. 6–11). These tables were formed by the aid of Scheutz's machine; a specialty of which is, that it records its results in the form of a mould to be employed as the matrix for a stereotype cast. It is consequently necessary, in the use of this machine, that each result, *as it arises*, should be correct in the last figure to be recorded. Dr. Farr informs us, in his Introduction (p. cxliii), that the method he employed for this purpose was that now under consideration. The required correction could probably, in this case, have been given in no other way.

Had the increase of 5 in the fifth place from the last not been made, the figures which now read 3,1, would have read 2,6; and in transcribing the result we should have had to correct it in the last place. The correction made in this manner once for all, suffices for the whole of the succeeding values; and, as has been already said, we shall not have occasion to look at any figures except those in the space between the pegs.

The expression (2) now comes into use for the completion of the series. It is,

$$A_{x+1}=A_x-(1-v)\Delta a_x;$$

which when $x=10$ becomes,

$$A_{11}=A_{10}-(1-v)\Delta a_{10}.$$

This also is of the form $P-QR$; but here R, that is Δa_{10}, being negative, the regulator must be set for addition. In other respects the setting of the machine remains; and after effacing (S_2), the series is completed by employing as multipliers the successive terms of Δa_x, effacing each, of course, after it has been used.

The series Δa_x should be formed on a separate slip of paper or cardboard (and proved by addition), as it comes into use in other formations.

The commencing portion of the results, as they come out, is here given; and it may be compared with the corresponding portion of the series* on p. 14 of the Institute volume.

x	Δa_x	A_x
10		·267523
11	1531	·271982
12	1811	·277257
13	2039	·283195
14	2206	·289621
15	2316	·296366
16	2366	·303258
17	2354	·310114
18	2284	·316766
19	2148	·323023
20	2004	·328860
	1941	
*	*	*

* The comparison here suggested will reveal the existence of a few slight discrepancies in the last place between the two sets of values compared. Both nevertheless are correct deductions from the data employed in their formation. The discrepancies, such as they are, originate in the relation which subsists between the values of an annuity and an assurance on the same status. It can easily be shown that, *at three per-cent*, four-decimal annuities are barely sufficient for the accurate determination of six-decimal assurances.

Although the machine (when in order) does not commit mistakes, the operator enjoys no such immunity from error. He may, for instance, have used a wrong figure in one of the multipliers;* and the error thus committed will, unless discovered and corrected, vitiate the succeeding values. It is, therefore, necessary to be provided with means for its speedy detection. These will be three or four terms of the series, at equal intervals, which may be readily formed in the same manner as the initial term. Each of these, as it is reached, will form a check on all the preceding work.

The series A_x may be formed with the same facility if we commence the formation with the oldest instead of the youngest tabular age. The chief difference in the working will be, that as in this order Δa_x is positive, the regulator will remain at subtraction throughout. The initial term will be

$$A_{97}=1-(1-v)(1+a_{97})$$
$$=1-(1-v)\times 1\cdot 0000,$$

since $A_{97}=v$; and it will be formed, after setting on P and Q as for A_{10}, by a single turn, *without pushing in the slide.*

Example 2.—Let the given annuity series be that in column 64, p. 150; to form the corresponding assurance series.

The process here is in such entire analogy to that exemplified in Example 1, that it will be sufficient, after two remarks, to give a specimen.

The first remark is, that while, as a rule, the differences of the annuities are negative, there are exceptions; as here $\Delta a_{64\cdot 10}$ is positive. Differences thus abnormally affected should be written in red ink; and this will serve as a warning, when using them, to alter the regulator from its normal position.

The second remark is, that, whereas the differences of the single-life annuities consist generally of four significant figures, those of the joint-life annuities, as here arranged, have never more than three. The assurances corresponding to these will therefore admit of construction at the cost of a proportionally less amount of labour. I find, in fact, that a column (45 values), after the differences and verifications have been formed, need not occupy in construction (including the recording), more than twenty minutes. It would hence be quite a practicable task to form the assurances

* The facility with which errors of this kind, *when noticed at the time*, admit of correction, forms one of the great merits of the Arithmometer. The erroneous figure being brought into the working position, it is set right by the requisite number of turns; remembering that, preparatory thereto, if it is diminution that is required, the regulator must be reversed.

corresponding to the joint-life and last-survivor annuities in the Institute volume.

The following is the specimen above referred to :—

x		
10	$\Delta a_{64.x}$	·723150
11	32	·723056
12	58	·723225
13	136	·723621
14	196	·724192
15	239	·724888
16	263	·725654
17	264	·726423
18	238	·727117
19	186	·727658
20	121	·728011
	87	
*	*	*

Problem II.—A table of annuities being given, it is required to construct a table of the Values of Policies at all ages, and for all durations.

The technical "Value of a Policy" is the difference between the value, on an anniversary of its inception (and consequently when a premium is just due), of the sum assured (supposed a unit) and the premium payable in respect of it, on the supposition that this is the *net premium* according to the table used in the valuation.

Denoting, for the present purpose, the value of a policy on a life now aged x, which was effected at age w, by $V_{w]x}$*, it is known that,

$$V_{w]x}=1-\frac{1+a_x}{1+a_w}$$

This may be written,

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x);$$

so that by using the reciprocal of $1+a_w$ the indicated division is changed into a multiplication. The expression is now of the form $P-QR$; and it is consequently fitted for the application of the Arithmometer.

The given annuity table being, as before, that on p. 14, the following scheme will facilitate the comprehension of the order of construction :—

* I find that the symbol for the value of a policy, ${}_nV_x$, given in the recognized notation, is unsuited when, as in the present investigation, it is necessary to treat the value as a function of the ages at entry and at valuation. I have therefore been obliged to devise a new symbol; and the one in the text, while answering my present purpose sufficiently well, is so distinctive that there is no risk of its being confused with any other symbol.

x	10	11	12	13	14	15	16
10	10·10						
11	11·10	11·11					
12	12·10	12·11	12·12				
13	13·10	13·11	13·12	13·13			
14	14·10	14·11	14·12	14·13	14·14		
15	15·10	15·11	15·12	15·13	15·14	15·15	
16	16·10	16·11	16·12	16·13	16·14	16·15	16·16

Here the values of x, the present age, are at the side, and those of w, the age at entry, at the top; so that w is constant in the columns, and x in the rows.

We have, $V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)$ . . . . . (3)

It is most convenient to effect the construction in columns. Hence, making x the variable,

$$\Delta_x V_{w]x}=-(1+a_w)^{-1}.\Delta a_x;$$

and, $$V_{w]x+1}=V_{w]x}-(1+a_w)^{-1}.\Delta a_x \quad . \quad . \quad . \quad (4)$$

The similarity of these expressions to those that arose in Problem I. is apparent; the only difference being that in these $(1+a_w)^{-1}$ takes the place of $1-v$ in the others. The process of construction, accordingly, in each column, after the formation of the initial term, is absolutely identical with that of Problem I.

The initial term for the first column is

$$V_{10]10}=1-(1+a_{10})^{-1}.(1+a_{10});$$

so that we have,

$$P= \quad 1·0000000,$$
$$Q=(1+a_{10})^{-1}= \quad ·0397640,$$
$$R=1+a_{10} \quad =25·1484;$$

and the setting of the machine will be as follows:—

```
0 0 0 0 1,0 0 0 0 0,5 0|0 0 0 0
                    *
          0 0 3 9 7 6 4 0|
```

From (3) it appears that when $x=w$, $V_{w]x}=0$. That is, the initial term in all the columns is 0; and we might therefore assume this value, and commence the construction in each column with the formula (4). It is better, however, to proceed regularly, as we should not otherwise have that perfect identity which ought to subsist, between the values successively formed and those formed for verification in the manner to be presently shown. The reciprocals are only approximately, not absolutely true; and $V_{w]x}$ when $x=w$

never comes out exactly equal to 0, although, using the reciprocals to six significant figures, the deviation will in no case affect the *sixth* place of our results by more than a unit.

The numbers being placed on the machine as above, pushing in the slide, setting the regulator for subtraction, and multiplying by 251484, we get for $V_{10]10}$,

$$0\,0\,0\,0\,0,0\,0\,0\,0\,0,\overset{*}{4}\,0\,2\,2\,4\,0.$$

The regulator is now changed to addition, Δa_x being negative throughout, and multiplication being made by the terms of this series in succession, commencing with $\Delta a_{10}=1531$, column 10 is completed.

The other columns are formed in the same way, the initial values of w and x in each succeeding column being respectively greater by unity than in the column preceding.

The following is a specimen of the formation, embracing the commencement of the first six columns.

		397640	400075	402995	406334	410009	413940
w		10	11	12	13	14	15
10	Δa_x	·00000					
11	1531	·00609	·00000				
12	1811	·01329	·00725	·00000			
13	2039	·02140	·01540	·00822	·00000		
14	2206	·03017	·02423	·01711	·00896	·00000	
15	2316	·03938	·03349	·02644	·01837	·00950	·00000
16	2366	·04879	·04296	·03598	·02799	·01920	·00979
17	2354	·05815	·05238	·04546	·03755	·02885	·01954
18	2284	·06723	·06152	·05467	·04683	·03821	·02899
19	2148	·07577	·07011	·06332	·05556	·04702	·03788
20	2004	·08374	·07813	·07140	·06370	·05524	·04618
	1941						
*	*	*	*	*	*	*	*

The reciprocals* that come into use in the formation of the several columns, being those of $1+a_{10}$, $1+a_{11}$, &c., are here, for illustration, written at the top of the respective columns; and the slip containing the series Δa_x is represented by the side of column 10. It may, as each column is completed, be moved forward to the next.

And now, as to the formation of verifications. Resuming the expression (3)

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)$$

if in this we make w the variable, the values indicated will be those

* The reciprocals for the entire annuity column had better be formed at the outset, either by the machine or by Oakes's Table.

occupying the row opposite x. We may therefore, by choosing three or four values of x at suitable intervals, and forming the corresponding horizontal series, thus obtain the requisite values for verification of the work in the columns. Taking the difference with respect to w, therefore, we have,

$$\Delta_w V_{w]x} = -(1+a_x).\Delta(1+a_w)^{-1};$$

whence, $$V_{w+1]x} = V_{w]x} - (1+a_x).\Delta(1+a_w)^{-1}. \quad . \quad . \quad (5)$$

Making $x=20$ and $w=10$, we have for the initial term of the row opposite 20,

$$V_{10]20} = 1-(1+a_{20})(1+a_{10})^{-1},$$

and for the next term

$$V_{11]20} = V_{10]20} - (1+a_{20}).\Delta(1+a_{10})^{-1};$$

the series being continued by using as multipliers the successive terms of the series $\Delta(1+a_w)^{-1}$, which is deduced from the values placed at the tops of the several columns in the last formation.

For the initial term:—

$$P = \qquad 1{\cdot}0000,$$
$$Q = 1+a_{20} = 23{\cdot}0425,$$
$$R = (1+a_{10})^{-1} = {\cdot}0397640;$$

and P and Q are placed on the machine as follows:—

```
0 0 0 0 1,0 0 0 0|0,5* 0 0 0 0 0
  0 0 2 3 0 4 2 5|
```

There being *seven* decimal places in R, the denominations in P and Q are made to correspond when the slide is drawn out *seven* holes.

Pushing in the slide, setting the regulator for subtraction, and multiplying successively by $(1+a_{10})^{-1}=397640$, and the differences of this series, the terms in line with $x=20$ come out as follows:—

w	$\Delta(1+a_w)^{-1}$	$V_{w]20}$
10		·08374
11	2435	·07813
12	2920	·07140
13	3339	·06370
14	3675	·05524
15	3931	·04618
16	4094	·03675
17	4156	·02717
18	4109	·01770
19	3940	·00862
20	3742	·00000
	3686	
*	*	*

The whole of the work in Columns 10 to 20, down to $x=20$, is thus verified. And in the same way verification may be obtained at as many points as we please, by forming the requisite horizontal series. These ought to be formed first, and inserted in their places, in order that, if error be committed in any of the columns, it may be arrested, and not suffered to proceed beyond the next point of verification.

It may be pointed out that the last horizontal series,—that corresponding to $x=97$,—which will serve as a final verification of all the columns, may be formed most conveniently without the aid of the machine, as follows:—When $x=97$, the expression for the value of the policy becomes,

$$V_{w]97}=1-(1+a_w)^{-1};$$

and the series of final terms will be formed in order, by subtracting from unity, continuously, $(1+a_{10})^{-1}$ and the differences of the series of which this is the first term. Thus:—

	$V_{w]97}$
	1·0000050 *
	0397640
10	·9602410
	2435
11	·9599975
	2920
12	·9597055
	3339
13	·9593716
	3675
14	·9590041
	3931
15	·9586110
	4094
16	·9582016
	4156
17	·9577860
*	*

The last term of this series ($x=97$, $w=97$), like those of all the other horizontal series, will come out equal to 0.

There is another arrangement of the table of the Values of Policies which is usually preferred. In it the values of w are still at the top; but the argument at the side, instead of x, is $x-w$, the duration of the policy. The following is the commencement of the table according to this arrangement.

$x-w$	10	11	12	13	14	15	$x-w$
0	·00000	·00000	·00000	·00000	·00000	·00000	0
1	·00609	·00725	·00822	·00896	·00950	·00979	1
2	·01329	·01540	·01711	·01837	·01920	·01954	2
3	·02140	·02423	·02644	.02799	·02885	·02899	3
4	·03017	·03349	·03598	·03755	·03821	·03788	4
5	·03938	·04296	·04546	·04683	·04702	·04618	5
6	·04879	·05238	·05467	·05556	·05524	·05421	6
7	·05815	·06152	·06332	·06370	·06320	·06217	7
8	·06723	·07011	·07140	·07159	·07108	·07031	8
9	·07577	·07813	·07922	·07940	·07914	·07882	9
10	·08374	·08589	·08697	·08739	·08757	·08776	10
*	*	*	*	*	*	*	*

The computation can be conducted in this form just as easily as in the other. The only changes in the process will be, that the slip containing the differences, while being carried forward from each column to the next, will have to be raised one line; and that the series formed for verification will take their places, not in horizontal lines, but in ascending diagonal lines.

PROBLEM III.—To construct Columns N_x and D_x, l_x and v^x being given.

It is advisedly that I here place N_x before D_x, the construction in this order being the easier of the two.

We have, $N_x = D_{x+1} + D_{x+2} + \dots;$

whence, $\Delta N_x = -D_{x+1} = -v^{x+1} l_{x+1},$

and, $N_{x+1} = N_x + \Delta N_x = N_x - v^{x+1} l_{x+1}.$

By aid of this expression, which is of the form $P-QR$, the column might be constructed, if we had the means of determining independently N_x for the youngest tabular age, namely 10 years. But this we have not. We therefore commence with the oldest age, and writing the above

$$N_x = N_{x+1} + v^{x+1} l_{x+1},$$

we have for an initial term, N_{97} being $=0$,

$$N_{96} = v^{97} l_{97}.$$

The method of forming this is obvious.

For the remainder of the column, the expression being now of the form, P+QR, we have $P=N_{x+1}$, $Q=v^{x+1}$, and $R=l_{x+1}$; and the manner of proceeding here, too, is obvious.

It will be observed, however, that the operation here differs in an important respect from those of the preceding problems. In them Q (the *in* factor) being constant, the setting on F remains unchanged till the completion of the column; here, however, $Q(=v^{x+1})$ varies with x, and the setting has to be altered for each term. The operation is nevertheless still continuous, inasmuch as each result enters into, and forms part of, that which follows.

The following is a specimen of the process. It includes also the formation of D_x.

	v^x			
	520 3284			
x	535 9383		N_x	D_x
97	552 0164	l_x	·00000000	·51171921
6	568 5769	9	·51171921	2·86960758
5	585 6342	49	3·38132679	8·14324320
4	603 2032	135	11·52456999	17·02360082
3	621 2993	274	28·54817081	30·01310627
2	639 9383	469	58·56127708	47·65556172
1	659 1364	723	106·21683880	71·42138460
0	678 9105	1052	177·63822340	102·09457340
*	* *	*	* * *	* * *
59	1697 3309	58866	112263·88510	10582·68657
8	1748 2508	60533	122846·57167	11186·83881
7	1800 6984	62125	134033·41048	11805·65929
6	1854 7193	63652	145839·06977	12439·12396
5	1910 3609	65114	158278·19373	13087·57478
4	1967 6717	66513	171365·76851	13751·57773
3	2026 7019	67852	185117·34624	14432·57755
2	2087 5029	69138	199549·92379	15131·09578
1	2150 1280	70373	214681·01957	15849·23394
0	2214 6318	71566	230530·25351	16589·31550
*	* *	*	* * *	* * *
19	5536 7575	96223	1227620·99017	55191·71170
8	5702 8603	96779	1282812·70187	57121·18885
7	5873 9461	97245	1339933·89072	59064·12591
6	6050 1645	97624	1398998·01663	61034·21644
5	6231 6694	97942	1460032·23307	63046·24818
4	6418 6195	98224	1523078·48125	65117·45981
3	6611 1781	98496	1588195·94106	67267·09717
2	6809 5134	98784	1655463·03823	69515·86405
1	7013 7988	99113	1724978·90228	71888·13958
0	7224 2126	99510	1796867·04186	74409·39100
	7440 9391	100000	1871276·43286	

The two columns following that containing the ages, represent two cardboard slips, on which are written, in reverse order, the terms of v^x and l_x, respectively. Their position with respect to each other and to the ages in the margin, is regulated in accordance with the formula of construction. Here we have, for example, opposite $x=96$, v^{97} and l_{97}; opposite $x=95$, v^{96} and l_{96}, and so on. The process now consists in the multiplying together of the corresponding numbers on the cards, in succession and the setting down of the results, as they arise, on the same line. These, as stated, are the terms of N_x corresponding to the values of x in the margin. It will be understood that in this process, in accordance with the formula of construction, the results are not removed from the machine, (S_2) alone requiring to be effaced after each multiplication.

I say nothing here as to the method of procuring verification of the foregoing process. I defer this, with further remarks on the construction, till the next following problem shall have been discussed.

Next, to form D_x. We have, as above,

$$D_x = -\Delta N_{x-1} = N_{x-1} - N_x.$$

Hence the Column D_x will be formed by differencing, in the usual way, the Column N_x, when written as above in reverse order. And the differences, when written as in the specimen, each opposite the subtrahend by the employment of which it is deduced, will be in their proper relation to the ages in the margin. And we have now, consequently, in line with x, v^{x+1}, l_{x+1}, N_x, and D_x.

To form D_x independently of N_x, the same process as that for the formation of the latter column would have to be gone through, with the addition that besides (S_2), (S_1) also would have to be effaced after each multiplication. The process also would be discontinuous; and in consequence, unless the work were performed in duplicate, no verification could be procured till both N_x and M_x should be in course of formation. If to this it be added that the deduction of D_x from N_x is at least as easy as that of N_x from D_x, sufficient will have been said to make it manifest that, in the construction of a Commutation Table by the aid of the Arithmometer, the proper course, in regard to the annuity columns, is to commence the formation with N_x. In the case of the assurance columns also, corresponding advantages attend the commencement of the formation with M_x.

PROBLEM IV.—Given v^x and d_x; to construct Columns M_x and C_x.

Since,
$$M_x = C_x + C_{x+1} + \dots$$

$\therefore$
$$\Delta M_x = -C_x = -v^{x+1} d_x .$$

Hence,
$$M_{x+1} = M_x + \Delta M_x = M_x - v^{x+1} d_x .$$

Column M_x, like Column N_x, and for a like reason, must be constructed in reverse order. Accordingly, transposing, we have,

$$M_x = M_{x+1} + v^{x+1} d_x ,$$

a formula for the construction in such entire accordance with that of the last problem, that little more is necessary than to direct attention to the specimen here given.

x	v^x	d_x	M_x	C_x
	520 3284			
	535 9383		·00000000	
97	552 0164	9	·49681476	·49681476
6	568 5769	40	2·77112236	2·27430760
5	585 6342	86	7·80757648	5·03645412
4	603 2032	139	16·19210096	8·38452448
3	621 2993	195	28·30743731	12·11533635
2	639 9383	254	44·56187013	16·25443282
1	659 1364	329	66·24745769	21·68558756
0	678 9105	408	93·94700609	27·69954840
*	* *	*	* * *	* * *
59	1697 3309	1667	7004·63125	282·94506
8	1748 2508	1592	7282·95278	278·32153
7	1800 6984	1527	7557·91943	274·96665
6	1854 7193	1462	7829·07939	271·15996
5	1910 3609	1399	8096·33888	267·25949
4	1967 6717	1339	8359·81012	263·47124
3	2026 7019	1286	8620·44398	260·63386
2	2087 5029	1235	8878·25059	257·80661
1	2150 1280	1193	9134·76086	256·51027
0	2214 6318	1160	9391·65815	256·89729
*	* *	*	* * *	* * *
19	5536 7575	556	17828·23462	307·84372
8	5702 8603	466	18093·98791	265·75329
7	5873 9461	379	18316·61047	222·62256
6	6050 1645	318	18509·00570	192·39523
5	6231 6694	282	18684·73878	175·73308
4	6418 6195	272	18859·32523	174·58645
3	6611 1781	288	19049·72716	190·40193
2	6809 5134	329	19273·76015	224·03299
1	7013 7988	397	19552·20796	278·44781
0	7224 2126	490	19906·19438	353·98642

The cards here are v^x and d_x, and their disposition, to accord with the working formula, is somewhat different from that of the cards in the last problem. Thus, in line with age x we have here v^{x+1} and d_x. When so arranged, the process for the formation of M_x is identical with that of last problem for the formation of N_x.

To form C_x,

$$C_x = -\Delta M_x = M_x - M_{x+1}.$$

Hence the column will be formed by differencing, as they stand, the terms of M_x; observing that here the differences as they arise are to be written, not as in the last problem, opposite the subtrahends, but opposite the minuends. When so written we have, in line with x, v^{x+1}, d_x, M_x, and C_x.

It has not been usual hitherto to tabulate C_x. But this column can be turned to such good account, for the formation of various subsidiary tables by the aid of the Arithmometer, that it will probably come to be tabulated, in a special category, along with others similarly adapted, such as Δa_x, D_x^{-1}, N_x^{-1}, &c.

When we multiply together two factors, of which one is a non-terminating number, a portion of the result has to be rejected, as not properly belonging to the product; since it would be altered if the non-terminating number were further extended. And therefore it is that in the process of contracted multiplication labour is saved by so arranging the work that only the *correct* portion of the product is formed. The Arithmometer attains the same end in another way: in the use of it we form the *entire* product of the numbers submitted to it, and we neglect the useless figures in recording the results. It is therefore desirable to be furnished with a guide as to the extent to which this process of curtailment ought to be carried.

In the case supposed—the multiplication of a terminating and a non-terminating factor—we know that we can depend upon about as many figures in the product as there are of significant figures in the non-terminating factor; and hence in the several terms of D_x and C_x, we shall have as many places true as there are in the terms of v^x which enter them, respectively. Also, in the several terms of N_x and M_x, which are respectively summations of series of terms in D_x and C_x, we should expect to find one or two more figures correct than in the corresponding terms of D_x and C_x.

We may say, then, that in the formation of N_x and M_x it is unnecessary to record any term to more than two places beyond the number of significant figures in the power of v in immediate use.

We shall find these conclusions verified in the case of the columns before us. I have used in the formation the column v^x (3 per-cent) as contained in Jones, vol. i, pp. 79 and 82, having first verified it by the aid of the Arithmometer. It contains seven significant figures at the outset, which further on are increased to eight; and I have used it to its full extent for experiment, although generally one or two places fewer will be considered sufficient.

Corresponding values in N_x and M_x are connected by the equation,

$$M_x = vN_{x-1} - N_x;$$

and applying this at a few points we find,

$$\begin{aligned} M_{91} &= \quad 66{\cdot}24745,576 \\ M_{59} &= \ 7004{\cdot}631,09 \\ M_{19} &= 17828{\cdot}234,95 \\ M_{10} &= 19906{\cdot}1939,3 \end{aligned}$$

The commas indicate the extent to which the values thus formed agree with those formed by the use of the machine. The agreement in the four examples extends to seven, eight, and nine places, respectively; but when we attend to the manner in which the formula lends itself to these deductions, we shall not fail to see that in each case the correctness of the Ns concerned is established generally to two places more. For example:—

103)128281270187	N_{18}
124544922512	vN_{18}
122762099017	N_{19}
17828234,95	M_{19}

Ten places in N_{18} and N_{19} we perceive here come into use for the determination of the eight correct places in M_{19}. We conclude, therefore, that at this part of the table N_x may be depended on to ten places of figures, and no more than this number need have been recorded.

Also, if the work has been correctly performed, there appears no reason why M_x should not be accepted as true to the same extent as N_x. As to the number of places to be finally tabulated for use, the computer will, of course, follow the dictates of his own judgment.

By the use of the foregoing formula, N_x and M_x may be verified as the work proceeds. Both columns being brought down to the same point by the formation of, say, twenty terms in each, comparison can be made at this point as above shown. If found satisfactory, the formation may be continued till another point of comparison is reached; and so on till the columns are completed.

It is hardly necessary to point out that the contrivance formerly suggested for correcting the last figure retained in the several terms, cannot in these constructions, be advantageously applied; the reason being, that here the last figures vary in their distances from the decimal point.

The pegs also are employed here in a manner different from that in which they came into use in Problems I and II. They are now employed to facilitate the recording, by separating the results on S_1 into convenient periods.

(To be continued.)

Note on a Method of finding the Value of an Annuity on the Last Surviver of Three Lives.

MR. William Godward, of the Law Life Office, has communicated to us a method by which the values of annuities on the last surviver of two lives contained in the Institute Life Tables may be very conveniently applied to calculate the values of annuities on the last surviver of three lives.

Let x, y, z be the three lives, of which x is the youngest, and let w be the single age equivalent to the joint lives y and z (so that $a_w = a_{yz}$), then the value of the annuity on the last surviver of x, y, z may be found by the formula

$$a_{\overline{xyz}} = a_{\overline{xy}} + a_{\overline{xz}} - a_{\overline{xw}}.$$

The truth of this formula, which Mr. Godward informs us is given by Simpson in the Supplement to his Annuities, p. 58, is easily demonstrated.